Uses and limitations of asymptotic analysis

Or: How to assume like a computer scientist

[AUT CSC]

AUT Computer Science Club

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Overview

- Recap and formalisms
- The usual suspects
- Limitations
- Questions

Recap and formalisms

Where we last left our intrepid heroes...

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- A model of a computer (RAM)
- Worst-case, asymptotic analysis

Example from last week

function $\text{Max}(\text{arr})$
    $\text{x} \leftarrow \text{arr}[1]$
    for $\text{i} \in 2, 3, \ldots, \text{length}(\text{arr})$ **do**
        if $\text{x} < \text{arr}[\text{i}]$ **then**
            $\text{x} \leftarrow \text{arr}[\text{i}]$
        end if
    end for
    return x
end function

Example from last week

function Max(arr)
  x ← arr[1]
  for i $\in 2, 3, \ldots,$ length(arr) **do**
    if x $<$ arr[i] **then**
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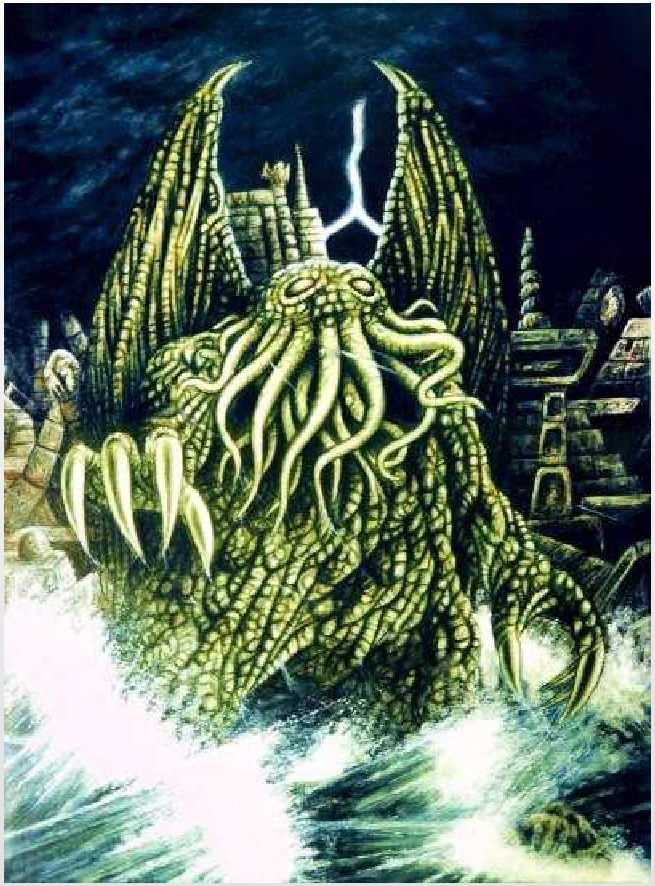
$S_{\mathsf{Max}}(n)$ is $O(1)$

What people see when they look at a formal definition of O



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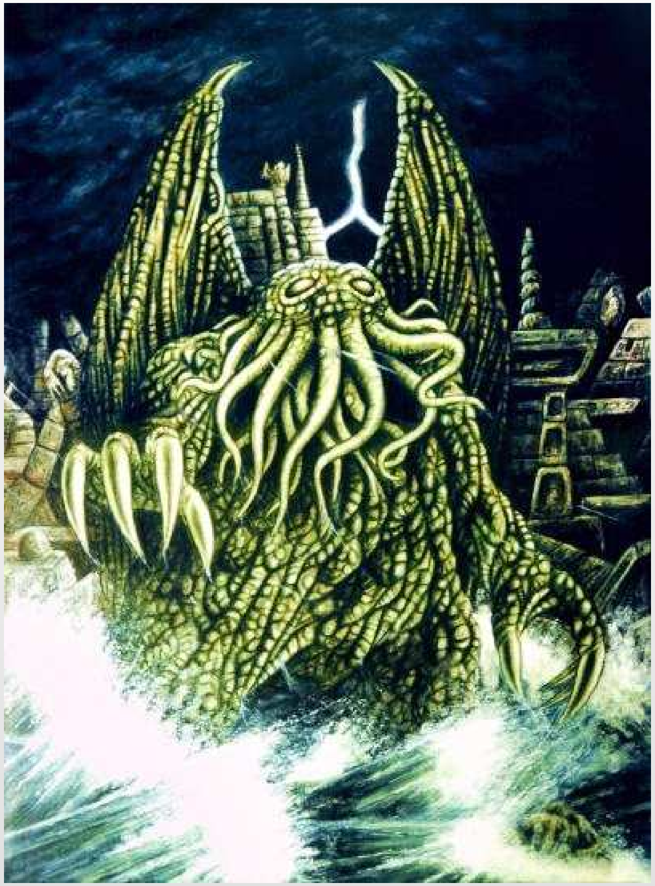

Don’t worry — we’ll go **slow**.

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We write $O(n)$ as a kind of shorthand — otherwise, we would have to write $O(f$ such that $f(n) = n)$, which is far too long.

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Thus, by definition, $6n - 1$ is $O(n)$. $\square$

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By substitution, we have $(c + 1)^2 \leq c \cdot (c + 1)$, which simplifies to $c + 1 \leq c$, which is a contradiction.

As both cases lead to contradictions, no such n_0, c can exist. Thus, n^2 is not $O(n)$. $\square$

The usual suspects

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$O(n^3)$

Cubic: when input size doubles, complexity increases by a factor of eight.

The ‘*un*acceptable’ ones

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Only gets *worse* from here — but we usually don’t bother at this point.

Limitations

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Is it really sensible to *always* be *uncompromisingly* pessimistic?

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Is it really sensible to analyze (and *design*) algorithms based on such an old view of our machines?

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This seems to fly in the face of why we're doing this in the first place. But where to draw the line?

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“Trust, but verify.”

Questions