Coursework III

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MATH40002: Analysis I

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Problem 1

Let A and B be non-empty subsets of $\mathbb{R}$ such that $a \leq b$ for all $a \in A$ and $b \in B$. Show that $\sup A \leq \inf B$ and that the equality holds if and only if for all $\varepsilon > 0$, there are $a \in A$ and $b \in B$ such that $b - a < \varepsilon$.

Solution.

Proof. For the first part, "$\sup A \leq \inf B$" , we will prove this by contradiction. Let's assume that

$$\sup A > \inf B \tag{1}$$

Therefore, we can always find a number $c > 0$, such that $\sup A = \inf B + c$, which is equivalent to $\inf B = \sup A - c$. By the definiton of *supreme* and the proposition we learnt in the autumn term of Analysis I, we have

$$\exists\, a_0 \in A, \text{ such that } a_0 > \inf B. \tag{2}$$

Now, by the definition given of the subsets A and B, we have

$$\forall a \in A,\ \forall b \in B \Longrightarrow a \leq b.$$

Therefore, for every $a \in A$, a is a lower bound of B. Hence, by the definiton of $infimum$, we have

$$\forall a \in A,\ a \leq \inf B.$$

which is a contradiction of (2). Therefore, by the axiom of Trichotomy, we have

$$\sup A \leq \inf B.$$

For the second part, "$\sup A = \inf B \Longleftrightarrow \forall \varepsilon > 0,\ \exists a \in A,\ \exists b \in B,$ such that $b - a < \varepsilon$" , we first prove the "$\Longrightarrow$" direction.

"$\Longrightarrow$":

We suppose that $\sup A = \inf B$ and let $\varepsilon > 0$ be arbitrary. Then, we can choose a $a \in A$, such that

$$\sup A - \frac{\varepsilon}{2} < a \leq \sup A \tag{3}$$

and $b \in B$, such that

$$\inf B \leq b < \inf B + \frac{\varepsilon}{2} \tag{4}$$

This is possible since $\sup A$ and $\inf B$ are the least upper bound and the greatest lower bound of A and B, respectively. Then, we combine (3) and (4), by the property of inequalities,

$$b - a < \inf B + \frac{\varepsilon}{2} - (\sup A - \frac{\varepsilon}{2}) = \inf B - \sup A + \varepsilon = \varepsilon \tag{5}$$

since $\sup A = \inf B$, and we have proved the "$\Longrightarrow$" direction.

"$\Longleftarrow$":

For this, we will prove this by contradiction. We suppose that $\forall \varepsilon > 0$, $\exists a \in A$, $\exists b \in B$, such that $b - a < \varepsilon$. We have already proved that $\sup A \leq \inf B$, so we can only assume that $\sup A < \inf B$. Then, we set

$$\varepsilon = \inf B - \sup A > 0$$

By the assumption, we can find $a \in A$ and $b \in B$, such that $b - a < \inf B - \sup A$. But this implies that

$$b < \inf B + a - \sup A. \tag{6}$$

By the definition of *supreme* and *infimum*, we have

$$a \leq \sup A \iff a - \sup A \leq 0 \tag{7}$$

$$b \geq \inf B \iff b - \inf B \geq 0 \tag{8}$$

We combine (6) and (7) by the property of inequlities, we have

$$b < \inf B + a - \sup A \leq \inf B + 0 = \inf B$$
$$\Downarrow$$
$$b < \inf B$$

But this contradicts to (8). Therefore, $\inf B$ could not be greater than $\sup A$, and we have proved the "$\Longleftarrow$" direction.

□

Problem 2

Using lower and upper sums, show that the function $t \mapsto t^2$ is integrable on $[0, x]$ for all $x > 0$ and that $\int_0^x t^2 dt = \frac{x^3}{3}$.

Solution.

Proof. To show that the function $t \mapsto t^2$ is integrable on $[0, x]$ for all $x > 0$, we need to show that for any $\varepsilon > 0$, there exists a partition P of $[0, x]$ such that the upper sum $U(f, P)$ and the lower sum $L(f, P)$ satisfy

$$U(t^2, P) - L(t^2, P) < \varepsilon$$

This is equivalent to showing that

$$\lim_{n\to\infty} U(t^2, P_n) = \lim_{n\to\infty} L(t^2, P_n)$$

where (P_n) is a sequence of partitions, i.e $P_n = \{t_0, t_1, \dots, t_n\}$. To find the upper and lower sums, we need to find the maximum and minimum values of t^2 on each subinterval in $[0, x]$. Since $\frac{d(t^2)}{dt} = 2t > 0 \,\text{for}\, \forall t \in [0, x]$, t^2 is an increasing function on $[0, x]$. Hence, the maximum value on each subinterval is attained at the right endpoint. So if we divide $[0, x]$ into n equal subintervals of length $\Delta t = x/n$, then for each $i = 1, 2, \dots, n$, we have

$$M_i = \sup\{t^2 : t \in [t_{i-1}, t_i]\} = (t_i)^2 = \left(\frac{ix}{n}\right)^2 \tag{9}$$

The minimum value on each subinterval is attained at the left endpoint. So for each $i = 1, 2, \dots, n$, we have

$$m_i = \inf\{t^2 : t \in [t_{i-1}, t_i]\}t^2 = (t_{i-1})^2 = \left(\frac{(i-1)x}{n}\right)^2 \tag{10}$$

Now, using (9) and (10), we can compute the upper and lower sums as follows. For the upper sum, we have

$$U(t^2, P_n) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) = \sum_{i=1}^{n} \left(\frac{ix}{n}\right)^2 \frac{x}{n} = \frac{x^3}{n^3} \sum_{i=1}^{n} i^2 \tag{11}$$

and for the lower sum, we have

$$L(t^2, P_n) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) = \sum_{i=1}^{n} \left(\frac{(i-1)x}{n}\right)^2 \frac{x}{n} = \frac{x^3}{n^3} \sum_{i=1}^{n} (i-1)^2 \tag{12}$$

Using some formulas for sums of squares, we can simplify (11) and (12) as:

$$U(t^2, P_n) = \frac{x^3}{n^3} \sum_{i=1}^{n} i^2 = \frac{x^3}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{x^3}{6} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \tag{13}$$

$$L(t^2, P_n) = \frac{x^3}{n^3} \sum_{i=1}^{n} (i-1)^2 = \frac{x^3}{n^3} \left(\frac{n(n-1)(2n-1)}{6}\right) = \frac{x^3}{6} \left(2 - \frac{3}{n} + \frac{1}{n^2}\right) \tag{14}$$

Now we can see that as n increases, both upper and lower sums converge to the same limit:

$$\lim_{n \to \infty} U(t^2, P_n) = \lim_{n \to \infty} L(t^2, P_n) = \frac{x^3}{3}$$

Therefore, the function $t \mapsto t^2$ is integrable on $[0, x]$ for all $x > 0$. Since the value of the limits of the upper and lower sums are the same as $\frac{x^3}{3}$, we can conclude that the definite integral of t^2 on $[0, x]$ is

$$\int_0^x t^2 dt = \frac{x^3}{3}$$

$\square$