Coursework II

CID number: You could never find it, ahhah

MATH40002: Analysis I

Imperial College London

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Problem 1

1. This question is shamelessly stolen from Example 6.2.9(b) of "Introduction to Real Analysis" by R. Bartle and D. Sherbert.

   (a) Prove that if $c \in (100, 105)$ then $10 < \sqrt{c} < 11$.

   (b) Use the Mean Value Theorem to show that

   $$\frac{5}{22} < \sqrt{105} - 10 < \frac{1}{4}$$

   (c) Can you improve this estimate?

Solution.

Proof. (a) *Proof.* We consider the function $f(x) = \sqrt{x}$ for $x \in \mathbb{R}^+$. We konw that $f(x)$ is differentiable for $x > 0$ from the lecture. The derivative of $f(x)$ is $f'(x) = \frac{1}{2\sqrt{x}}$, which is positive for all $x \in \mathbb{R}^+$. Hence, $f(x)$ is strictly monotone incresing for $x \in (99, 122)$. Therefore, for $c \in (100, 105)$, $\sqrt{100} < \sqrt{c} < \sqrt{105} < \sqrt{121}$, which shows that

$$10 < \sqrt{c} < 11$$

□

(b) We still consider the function $f(x) = \sqrt{x}$ for $x \in \mathbb{R}^+$. The Mean Value Theorem states that:
Let $f : [a, b] \mapsto \mathbb{R}$ be continous on $[a, b]$ and differentiable on (a, b). Then, there is $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
We apply the Mean Value Theorem with $f(x) = \sqrt{x}$ and set $a = 100,\ b = 105$. From the lectre, we know that $f(x)$ is continous and differentiable for $x > 0$, $f(x)$ is continous on $[100, 105]$ and differentiable on $(100, 105)$. The derivative of $f(x)$ is $f'(x) = \frac{1}{2\sqrt{x}}$ for $x > 0$. Hence, by the Mean Value Theorem, we abtain that there exists a $c \in (100, 105)$ such that

$$\begin{aligned}
\frac{1}{2\sqrt{c}} &= \frac{\sqrt{105} - \sqrt{100}}{105 - 100} \\
\sqrt{105} - \sqrt{100} &= \frac{5}{2\sqrt{c}} \\
\sqrt{105} - 10 &= \frac{5}{2\sqrt{c}}
\end{aligned}$$

By (a), we know that $10 < \sqrt{c} < 11$, we can assert that

$$\begin{aligned}
\frac{5}{2 \times 11} &< \sqrt{105} - 10 < \frac{5}{2 \times 10} \\
\frac{5}{22} &< \sqrt{105} - 10 < \frac{1}{4}
\end{aligned}$$

by the property of the inequality.

(c) We can improve the estimate by using our conclusiton of (b). As $\frac{5}{22} < \sqrt{105} - 10 < \frac{1}{4}$, it follows that $10.2272 < \sqrt{105} < 10.2500$. Since $c \in (100, 105)$, $\sqrt{c} < \sqrt{105} < 10.2500$. Thus, we can easyily get that

$$\sqrt{105} - 10 = \frac{5}{2\sqrt{c}} > \frac{5}{2 \times 10.2500} > 0.2439$$

by the property of the inequality.
The improved estimate is $0.2439 < \sqrt{105} - 10 < 0.2500$.

$\square$