Coursework I

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MATH40002: Analysis I

Imperial College London

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Problem 1

Prove that if f and g are continuous functions such that $f(q) = g(q)$ for all $q \in \mathbb{Q}$, then $f = g$.

Solution.

Proof. We prove this by using the sequential continuity. It states that if f is continuous at x, then for any sequence $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} f(x_n) = f(x)$.

Let x be any real number. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can always find a sequence of rational numbers $\{q_n\} \subset \mathbb{Q}$, which satisfies,

$$\lim_{n\to\infty} q_n = x$$

for any real number x.

As $f(q) = g(q)$ for all $q \in \mathbb{Q}$, we have $f(q_n) = g(q_n)$ for any $n \in \mathbb{N}$. It follows that

$$\lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} g(q_n)$$

Since f and g are continuous functions, then they are continuous at any $x \in \mathbb{R}$ by the definition of continuous function. Then, by the sequential continuity we stated before and the equation (1), we have

$$\lim_{n\to\infty} f(q_n) = f(x).$$

$$\lim_{n\to\infty} g(q_n) = g(x).$$

By the equations (2), (3), (4), we immediately get this:

$$f(x) = g(x) \forall x \in \mathbb{R}$$

It means that $f = g$. □

Problem 2

Prove that the finite union of bounded sets is bounded.

Solution.

Proof. Suppose we have a collection of finite bounded sets, which is $\{S_1, S_2, S_3, \ldots, S_n\}$. Each bounded set has its own upper bound, which is $\{M_1, M_2, M_3, \ldots, M_n\}$ respectively and has its own lower bound, which is $\{m_1, m_2, m_3, \ldots, m_n\}$ respectively. We claim that the finite union of these sets $U_{i=1}^n S_i$ is upper bounded by $\max\{M_1, M_2, M_3, \ldots, M_n\}$ and lower bounded by $\min\{m_1, m_2, m_3 \ldots, m_n\}$.

Since any element $x \in U_{i=1}^{n} S_i$ must be in at least one of the bounded sets $\{S_1, S_2, S_3, \ldots, S_n\}$, then for any $x \in U_{i=1}^{n} S_i$, we have $x \leq \max\{M_1, M_2, M_3, \ldots, M_n\}$ and $x \geq \min\{m_1, m_2, m_3 \ldots, m_n\}$. Therefore, the finite union of bounded sets is also bounded. $\square$