Coursework II

CID number: Trust me, you don't want to know it.

MATH40007: Introduction to Applied Mathematics , 2023

Imperial College London

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Problem 1

Part I: For any integer $n \geq 0$ define

$$I_+(n) \equiv \int_0^1 e^y \sin(n\pi y)dy, \quad I_-(n) \equiv \int_0^1 e^{-y} \sin(n\pi y)dy.$$

(i) Calculate these two integrals explicitly.

(ii) Use the result of part (i) to find the Fourier sine series of both $\sinh y$ and $\cosh y$ over the interval $[0,1]$ (you should use ideas from the "Calculus and Applications" course).

Part II: Consider the electric circuit shown in the Figure where the vertical edges have conductance c and the horizontal edges have conductance d. Node 2 N + 1 is set to unit voltage, while nodes 0 and N + 1 to 2 N are grounded (set to zero voltage). Kirchhoff's current law holds at nodes 1 to N. Let $\hat{\mathbf{x}}$ denote the voltages at nodes 1 to N. The nodes should be ordered as follows: $1, 2, \ldots, 2\text{ N} - 1, 2\text{ N}, 0, 2\text{ N} + 1$.

(a) Show that the conductance-weighted Laplacian matrix is

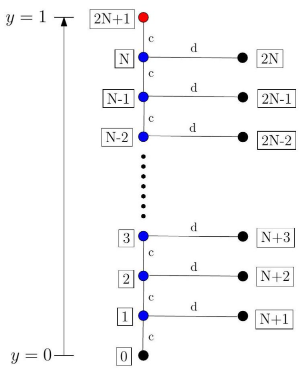
$$\mathbf{K} = \begin{pmatrix} c\mathbf{K}_N + d\mathbf{I}_N & -d\mathbf{I}_N & -c\mathbf{P} \\ -d\mathbf{I}_N & d\mathbf{I}_N & \mathbf{0} \\ -c\mathbf{P}^T & \mathbf{0} & c\mathbf{I}_2 \end{pmatrix},$$

where $\mathbf{I}_j$ denotes the j-by- j identity matrix and $\mathbf{K}_N$ is the N-by- N matrix familiar from lectures. You should find the N-by-2 matrix $\mathbf{P}$.

(b) Let $\{\mathbf{\Phi_j} \mid j = 1, \ldots, N\}$ and $\{\lambda_j \mid j = 1, \ldots, N\}$ denote the orthonormal eigenvectors and corresponding eigenvalues of $\mathbf{K}_N$. By writing

$$\hat{\mathbf{x}} = \sum_{j=1}^{N} a_j(\mu)\mathbf{\Phi}_j, \quad \mu = \frac{d}{c}$$

find the coefficients $\{a_j(\mu) \mid j = 1, \ldots, N\}$.



(c) Show that the n-th element of $\hat{\mathbf{x}}$ can also be written as

$$\frac{\lambda_+(\mu)^n - \lambda_-(\mu)^n}{\lambda_+(\mu)^{N+1} - \lambda_-(\mu)^{N+1}}, \quad n = 1, \ldots, N,$$

for suitable choices of the parameters $\lambda_\pm(\mu)$.

(d) The uniqueness theorem for harmonic potentials discussed in lectures has an analogous version when the conductances are not all equal. Use this fact to establish a discrete identity involving your answers to parts (b) and (c).

(e) Now pick μ to be given by

$$\mu = \frac{1}{(N+1)^2}$$

and introduce the new variable

$$y = \frac{n}{(N+1)}$$

Find the limit of both left- and right-hand sides of the discrete identity you found in part (d) as $N \to \infty$ with y taken to be fixed.

Solution.

Part I: (i)

$$\begin{aligned}
I_+(n) &= \int_0^1 e^y \sin(n\pi y)dy \\
&= \int_0^1 sin(n\pi y) d(e^y) \\
&= \sin(n\pi y)e^y\Big|_0^1 - n\pi \int_0^1 e^y \cos(n\pi y)dy \\
&= -n\pi \int_0^1 e^y \cos(n\pi y)dy \\
&= -n\pi \int_0^1 \cos(n\pi y)d(e^y) \\
&= -n\pi \cos(n\pi y)e^y\Big|_0^1 - n^2\pi^2 \int_0^1 e^y \sin(n\pi y)dy \\
&= -n\pi(\cos(n\pi)e - 1) - n^2\pi^2 I_+(n) \\
(n^2\pi^2 + 1)I_+(n) &= n\pi(1 - (-1)^n e) \\
I_+(n) &= \frac{n\pi(1 - (-1)^n e)}{n^2\pi^2 + 1} \\
I_-(n) &= \int_0^1 e^{-y} \sin(n\pi y)dy \\
&= -\int_0^1 \sin(n\pi y)d(e^{-y}) \\
&= -e^{-y}\sin(n\pi y)\Big|_0^1 + n\pi \int_0^1 e^{-y}\cos(n\pi y)dy \\
&= -n\pi \int_0^1 \cos(n\pi y)d(e^{-y})
\end{aligned}$$

$$
\begin{aligned}
&= -n\pi\cos(n\pi y)e^{-y}\Big|_0^1 - n^2\pi^2\int_0^1 e^{-y}\sin(n\pi y)dy \\
&= -n\pi(\cos(n\pi)e^{-1}-1) - n^2\pi^2 I_-(n) \\
I_-(n) + n^2\pi^2 I_-(n) &= n\pi(1-(-1)^n e^{-1}) \\
I_-(n) &= \frac{n\pi(1-(-1)^n e^{-1})}{n^2\pi^2+1}
\end{aligned}
$$

As conclusion, we have

$$
\begin{aligned}
I_+(n) &= \frac{n\pi(1-(-1)^n e)}{n^2\pi^2+1} \\
I_-(n) &= \frac{n\pi(1-(-1)^n e^{-1})}{n^2\pi^2+1}
\end{aligned}
$$

(ii) As $\sinh y$ is an odd function, we have $a_n = 0$ for all n. Therefore, at the interval $[0,1]$, we have

$$
\begin{aligned}
b_n &= 2\int_0^1 \frac{e^y - e^{-y}}{2}\sin(n\pi y)dy \\
&= \int e^y \sin(n\pi y)dy - \int_0^1 e^{-y}\sin(n\pi y)dy \\
&= I_+(n) - I_-(n)
\end{aligned}
$$

Hence, the Fourier sine series of $\sinh y$ is

$$
\sinh y = \sum_{n=1}^{\infty} b_n \sin(n\pi y) = \sum_{n=1}^{\infty}(I_+(n) - I_-(n))\sin(n\pi y) = \sum_{n=1}^{\infty}\frac{n\pi(-1)^n(e^{-1}-e)}{n^2\pi+1}\sin(n\pi y)
$$

For $\cosh y$, it is an even function. We can do the odd extension of $\cosh y$ to get the Fourier sine series of $\cosh y$. Hence,

$$
\begin{aligned}
b_n &= 2\int_0^1 \frac{e^y + e^{-y}}{2}\sin(n\pi y)dy \\
&= \int_0^1 e^y \sin(n\pi y)dy + \int_0^1 e^{-y}\sin(n\pi y)dy \\
&= I_+(n) + I_-(n)
\end{aligned}
$$

Hence, the Fourier sine series of $\cosh y$ is

$$
\cosh y = \sum_{n=1}^{\infty} b_n \sin(n\pi y) = \sum_{n=1}^{\infty}(I_+(n) + I_-(n))\sin(n\pi y) = \sum_{n=1}^{n}\frac{2n\pi(-1)^n(e^{-1}+e)}{n^2\pi+1}\sin(n\pi y)
$$

As conclusion, we have

$$
\sinh y = \sum_{n=1}^{\infty}\frac{n\pi(-1)^n(e^{-1}-e)}{n^2\pi+1}\sin(n\pi y)
$$

$$\cosh y = \sum_{n=1}^{n} \frac{2n\pi(-1)^n(e^{-1}+e)}{n^2\pi+1}\sin(n\pi y)$$

Part II: (a) By the order given, The conductance-weighted Laplacian matrix of the graph is given by

$$\mathbf{K} = \left[\begin{array}{cccccc|cccc|cc}
2c+d & -c & 0 & \cdots & 0 & 0 & -d & \cdots & 0 & 0 & -c & 0 \\
-c & 2c+d & -c & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -c & 2c+d & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2c+d & -c & 0 & \cdots & -d & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & -c & 2c+d & 0 & \cdots & 0 & -d & 0 & -c \\
\hline
-d & 0 & 0 & \cdots & 0 & 0 & d & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -d & 0 & 0 & \cdots & d & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -d & 0 & \cdots & 0 & d & 0 & 0 \\
\hline
-c & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & c & 0 \\
0 & 0 & 0 & \cdots & 0 & -c & 0 & \cdots & 0 & 0 & 0 & c
\end{array}\right]$$

which is equal to

$$\mathbf{K} = \begin{bmatrix} c\mathbf{K}_N + d\mathbf{I}_N & -d\mathbf{I}_N & -c\mathbf{P} \\ -d\mathbf{I}_N & d\mathbf{I}_N & \mathbf{0} \\ -c\mathbf{P}^T & \mathbf{0} & c\mathbf{I}_2 \end{bmatrix},$$

where $\mathbf{I}_j$ denotes the j-by- j identity matrix and $\mathbf{K}_N$ is the N-by- N matrix familiar from lectures and the N-by-N matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) For this electric circuit, we have:

$$\mathbf{K}\mathbf{X} = \mathbf{f} \tag{1}$$

$$\begin{bmatrix} c\mathbf{K}_N + d\mathbf{I}_N & -d\mathbf{I}_N & -c\mathbf{P} \\ -d\mathbf{I}_N & d\mathbf{I}_N & \mathbf{0} \\ -c\mathbf{P}^T & \mathbf{0} & c\mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \mathbf{0} \\ \hat{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{C_{eff}} \\ \hat{\mathbf{f}} \end{bmatrix} \tag{2}$$

where $\hat{\mathbf{x}}$ is the vector of the voltages at the nodes 1 to N. Since the nodes at $N+1$ to $2N$ are grounded, the voltages there are all 0 and $\hat{\mathbf{e}}$ is the vector of the voltages at the voltage source $2N+1$ and 0, $\mathbf{C_{eff}}$ is the vector of the effective conductance, and $\hat{\mathbf{f}}$ is the vector of the applied voltages. As KCL holds at nodes

1 to N, the flux of nodes 1 to N are all zero. In ditails, we have

$$\hat{\mathbf{e}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ \hat{\mathbf{f}} = \begin{bmatrix} -f_0 \\ f_0 \end{bmatrix}$$

The linear system (2) is equivalent to

$$c\mathbf{K}_N\hat{\mathbf{x}} + d\mathbf{I}_N\hat{\mathbf{x}} - d\mathbf{I}_N\mathbf{0} - c\mathbf{P}\hat{\mathbf{e}} = \mathbf{0} \tag{3}$$

$$d\mathbf{I}_N\hat{\mathbf{x}} - d\mathbf{I}_N\mathbf{0} = \mathbf{C_{eff}} \tag{4}$$

$$-c\mathbf{P}^T\hat{\mathbf{e}} = \hat{\mathbf{f}} \tag{5}$$

Let's consider equation (3), it impies that

$$c\mathbf{K}_N\hat{\mathbf{x}} + d\mathbf{I}_N\hat{\mathbf{x}} = c\mathbf{P}\hat{\mathbf{e}} \tag{6}$$

$$c\mathbf{K}_N\hat{\mathbf{x}} + d\hat{\mathbf{x}} = c\mathbf{P}\hat{\mathbf{e}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ c \end{bmatrix} \tag{7}$$

Let us solve equation (7) using the eigenvectors of $\mathbf{K}_N$, which we learnt in the lecture.

$$\mathbf{K}_N\mathbf{\Phi}_j = \lambda\mathbf{\Phi}_j,\ j = 1, 2, \cdots, N \tag{8}$$

where

$$\mathbf{\Phi}_j = \sqrt{\frac{2}{N+1}} \begin{pmatrix} \sin\left(\frac{j\pi}{N+1}\right) \\ \sin\left(\frac{2j\pi}{N+1}\right) \\ . \\ . \\ \sin\left(\frac{nj\pi}{N+1}\right) \end{pmatrix}, \quad j = 1, \ldots, N$$

which has corresponding eigenvalue

$$\lambda_j = 2 - 2\cos\left(\frac{\pi j}{N+1}\right), \quad j = 1, \ldots, N.$$

This orthonormal set of vectors can be used as a basis of the solution space. As

$$\hat{\mathbf{x}} = \sum_{j=1}^{N} a_j(\mu)\mathbf{\Phi}_j, \quad \mu = \frac{d}{c}$$

for some set of coefficients $\{a_j(\mu) \mid j = 1, \ldots, N\}$ to be determined. The equation (7) now tells us that

$$c\mathbf{K}_N\hat{\mathbf{x}} + d\hat{\mathbf{x}} = c\mathbf{K}_N(\sum_{j=1}^{N} a_j(\mu)\mathbf{\Phi}_j) \tag{9}$$

$$= c\sum_{j=1}^{N} a_j(\mu)\lambda_j \mathbf{\Phi}_j + d\sum_{j=1}^{N} a_j(\mu)\mathbf{\Phi}_j \tag{10}$$

$$= \sum_{j=1}^{N}(ca_j(\mu)\lambda_j + da_j(\mu))\mathbf{\Phi}_j = \sum_{j=1}^{N} a_j(\mu)(c\lambda_j + d)\mathbf{\Phi}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ c \end{bmatrix} \tag{11}$$

The orthonormality of the eigenvectors can be exploited to find the coefficients $a_j(\mu)$. To see this, note that on multiplying (11) by $\mathbf{\Phi}_j^T$, it follows that

$$\sum_{j=1}^{N} a_j(\mu)(c\lambda_j + d)\mathbf{\Phi}_m^T \mathbf{\Phi}_j = \mathbf{\Phi}_m^T \begin{bmatrix} 0 \\ 0 \\ \vdots \\ c \end{bmatrix} = c\sqrt{\frac{2}{N+1}}\sin(\frac{Nm\pi}{N+1}).$$

By the orthonormality of the eigenvectors, we have

$$\mathbf{\Phi}_m^T \mathbf{\Phi}_j = \delta_{mj}.$$

where δ_{mj} is the Kronecker delta. Therefore, we have

$$a_m(\mu)(c\lambda_m + d) = c\sqrt{\frac{2}{N+1}}\sin(\frac{Nm\pi}{N+1})$$
$$a_m(\mu) = \frac{c\sqrt{\frac{2}{N+1}}\sin(\frac{Nm\pi}{N+1})}{c\lambda_m + d}$$

As $\mu = \frac{d}{c}$, we have

$$a_m(\mu) = \sqrt{\frac{2}{N+1}}\frac{\sin(\frac{Nm\pi}{N+1})}{\lambda_m + \mu} = \sqrt{\frac{2}{N+1}}\frac{\sin(\frac{Nm\pi}{N+1})}{(2 - 2\cos(\frac{Nm\pi}{N+1})) + \mu}$$

Therefore, we have the coefficients $\{a_j(\mu) | j = 1, ..., N\}$ as

$$a_j(\mu) = \sqrt{\frac{2}{N+1}}\frac{\sin(\frac{Nj\pi}{N+1})}{(2 - 2\cos(\frac{Nj\pi}{N+1})) + \mu}$$

(c) As KCL holds at nodes 1 to N and node $2N + 1$ is set to unit voltage and node 0 is grounded, we have

$$x_0 = 1, \quad , x_{2N+1} = 1$$

For $n = 1, 2, ..., N$

$$c(x_{n+1} - x_n) = dx_n + c(x_{n-1} - x_n)$$
$$x_n = \frac{c}{2c + d}(x_{n+1} + x_{n-1})$$

$$= \frac{x_{n+1} + x_{n-1}}{\mu + 2}$$

where $\mu = \frac{d}{c}$. Therefore, we have

$$(2 + \mu)x_n = x_{n-1} + x_{n+1}, \quad n = 1, 2, ..., N$$

We get such a recrussion relation and it is linear, then we can solve it like this. We can transfer the relation into a characteristic equation,

$$(2 + \mu)\lambda^n = \lambda^{n-1} + \lambda^{n+1} \tag{12}$$

$$\lambda^{n+1} - (2 + \mu)\lambda^n + \lambda^{n-1} = 0 \tag{13}$$

$$\lambda^{n-1}(\lambda^2 - (2 + \mu)\lambda + 1) = 0 \tag{14}$$

As $n = 1, 2, ..., N$, $\lambda^{n-1} \neq 0$, then it must have $\lambda^2 - (2 + \mu)\lambda + 1 = 0$, then we can get the solution of λ,

$$\lambda = \frac{1}{2}(\mu + 2 \pm \sqrt{4\mu + \mu^2})$$

Therefore, we have

$$x_n = \frac{(\frac{1}{2}(\mu + 2 + \sqrt{4\mu + \mu^2}))^n - (\frac{1}{2}(\mu + 2 - \sqrt{4\mu + \mu^2}))^n}{(\frac{1}{2}(\mu + 2 + \sqrt{4\mu + \mu^2}))^{N+1} - (\frac{1}{2}(\mu + 2 - \sqrt{4\mu + \mu^2}))^{N+1}}$$

We set

$$\lambda_+(\mu) = \frac{1}{2}(\mu + 2 + \sqrt{4\mu + \mu^2})$$

$$\lambda_-(\mu) = \frac{1}{2}(\mu + 2 - \sqrt{4\mu + \mu^2}),$$

Then, we have

$$x_n = \frac{\lambda_+(\mu)^n - \lambda_-(\mu)^n}{\lambda_+(\mu)^{N+1} - \lambda_-(\mu)^{N+1}} \quad n = 1, 2, ..., N$$

(d) By the uniqueness theorem of hormonic potentials, the results from (b) and (c) must be equal. Therefore, we have

$$\frac{\lambda_+(\mu)^n - \lambda_-(\mu)^n}{\lambda_+(\mu)^{N+1} - \lambda_-(\mu)^{N+1}} = \sum_{j=1}^{N} \frac{2}{N+1} \frac{\sin(\frac{Nj\pi}{N+1})}{(2 - 2\cos(\frac{j\pi}{N+1})) + \mu} \sin(\frac{nj\pi}{N+1}) \tag{15}$$

$$\frac{\lambda_+(\mu)^n - \lambda_-(\mu)^n}{\lambda_+(\mu)^{N+1} - \lambda_-(\mu)^{N+1}} = \frac{2}{N+1} \sum_{j=1}^{N} \frac{\sin(\frac{Nj\pi}{N+1})}{(2 - 2\cos(\frac{j\pi}{N+1})) + \mu} \sin(\frac{nj\pi}{N+1}) \tag{16}$$

which is the discrete identity.

(e) We take the limit $N \to \infty$ at the both sides of identity (16) and use

$$\mu = \frac{1}{(N+1)^2}, \quad y = \frac{n}{N+1}.$$

then we get,

$$\lim_{N\to\infty} \frac{(\frac{1}{2}(\mu+2+\sqrt{4\mu+\mu^2}))^n - (\frac{1}{2}(\mu+2-\sqrt{4\mu+\mu^2}))^n}{(\frac{1}{2}(\mu+2+\sqrt{4\mu+\mu^2}))^{N+1} - (\frac{1}{2}(\mu+2-\sqrt{4\mu+\mu^2}))^{N+1}}$$
$$= \lim_{N\to\infty}\sum_{j=1}^{\infty} \frac{2}{N+1} \frac{\sin(\frac{Nj\pi}{N+1})}{(2-2\cos(\frac{j\pi}{N+1})) + \frac{1}{(N+1)^2}} \sin(j\pi y)$$

As $N \to \infty$, the $\frac{j\pi}{N+1}$ is very small and we use the Taylor series,

$$2-2\cos(\frac{j\pi}{N+1}) = 2(1-\cos(\frac{j\pi}{N+1})) = 2(1-(1-\frac{1}{2!}\frac{j^2\pi^2}{(N+1)^2})+\ldots) = \frac{j^2\pi^2}{(N+1)^2}+\ldots$$
$$\sin(\frac{j\pi}{N+1}) = \frac{\pi j}{N+1}+\ldots$$

Let us see the limitation again. From the Calculus and Applications course, we know that,

$$\lim_{N\to\infty}\left(\frac{1}{2}\left(\left(\frac{1}{(N+1)^2}\right)+2+\sqrt{4\frac{1}{(N+1)^2}+\frac{1}{(N+1)^2}}\right)\right)^n -$$
$$\left(\frac{1}{2}\left(\left(\frac{1}{(N+1)^2}\right)+2-\sqrt{4\frac{1}{(N+1)^2}+\frac{1}{(N+1)^2}}\right)\right)^n = 0$$

$$\lim_{N\to\infty}\left(\frac{1}{2}\left(\left(\frac{1}{(N+1)^2}\right)+2+\sqrt{4\frac{1}{(N+1)^2}+\frac{1}{(N+1)^2}}\right)\right)^{N+1} -$$
$$\left(\frac{1}{2}\left(\left(\frac{1}{(N+1)^2}\right)+2-\sqrt{4\frac{1}{(N+1)^2}+\frac{1}{(N+1)^2}}\right)\right)^{N+1} = e-\frac{1}{e}$$

For the identity, use the Taylor expansion of $2-2\cos(\frac{j\pi}{N+1})$,

$$\frac{0}{e-e^{-1}} = \lim_{N\to\infty}\sum_{j=1}^{\infty}\frac{2(N+1)\sin(\frac{Nj\pi}{N+1})}{j^2\pi^2+1}\sin(j\pi y)$$
$$0 = \lim_{N\to\infty}\sum_{j=1}^{\infty}\frac{2(N+1)(e-e^{-1})\sin(\frac{j\pi(N+1)-j\pi}{N+1})}{j^2\pi^2+1}\sin(j\pi y)$$
$$0 = \lim_{N\to\infty}\sum_{j=1}^{\infty}\frac{2(N+1)(e-e^{-1})(\sin(j\pi)\cos(\frac{j\pi}{N+1})-\cos(j\pi)\sin(\frac{j\pi}{N+1}))}{j^2\pi^2+1}\sin(j\pi y)$$
$$0 = \lim_{N\to\infty}\sum_{j=1}^{\infty}\frac{2(N+1)(e-e^{-1})(-(-1)^j\sin(\frac{j\pi}{N+1}))}{j^2\pi^2+1}\sin(j\pi y)$$

We use the Taylor expansion of $\sin(\frac{j\pi}{N+1})$,

$$
\begin{aligned}
0 &= \lim_{N\to\infty}\sum_{j=1}^{\infty}\frac{2(N+1)(e-e^{-1})(-(-1)^j\frac{\pi j}{N+1})}{j^2\pi^2+1}\sin(j\pi y)\\
0 &= 2\sum_{j=1}^{\infty}\frac{j\pi(e^{-1}-e)(-1)^j}{j^2\pi^2+1}\sin(j\pi y)\\
0 &= \sum_{j=1}^{\infty}\frac{j\pi(e^{-1}-e)(-1)^j}{j^2\pi^2+1}\sin(j\pi y)
\end{aligned}
$$

We can observe that

$$\sum_{j=1}^{\infty}\frac{j\pi(e^{-1}-e)(-1)^j}{j^2\pi^2+1}\sin(j\pi y)$$

is the Fourier sine series of $\sinh y$ from Part I.
Since

$$
\begin{aligned}
y = \frac{n}{N+1} \to 0 \quad when \quad N\to\infty\\
\sinh y = 0 \quad when \quad y = 0
\end{aligned}
$$

We have

$$\sum_{j=1}^{\infty}\frac{j\pi(e^{-1}-e)(-1)^j}{j^2\pi^2+1}\sin(j\pi y) = \sinh 0 = 0$$

Therefore, the value of $\sinh y$ when $y = 0$ is coincide with what we calculate in (e). It is clear that the Fourier sine series of $\sinh y$ is zero when y is fixed as $\frac{n}{1+N}$.