# Dependency graphs

## Or: why scheduling is *hard*

Koz Ross

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**[AUT CSC]**

**AUT Computer Science Club**

# Outline

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- Other tasks *must* be done in a certain order (e.g. $2$ must happen before $3$)
- These together determine how (and how fast!) we can finish all the tasks

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- *Especially* important for computers (parallel processing is how we get all of our speed gains these days)
- A special (and limited) case of *scheduling* (this acts as a 'difficulty floor' for the more general version)

# The basics

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For example,
$P(\mathbb{N}_3) = \{\{\}, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$.

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- Take a dump (numbered $1$)
- Brush teeth (numbered $2$)

A possible $3$-task list for that would be $\{(0, \{\}), (1, \{0\}), (2, \{0\})\}$.

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We have a *cyclic dependency between* $x$ *and* $y$ *in* $T_k$ if $x$ is a transitive dependency of $y$ and $y$ is a transitive dependency of $x$.

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Where clear, we will drop references to $T_k$ from now on.

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The *critical path of* $T_k$ is the longest dependency chain.

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5. How efficiently can we compute all of these things?

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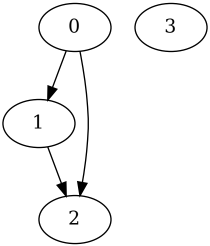


This is a visual representation of the graph $(\mathbb{N}_4, \{(0,1), (0,2), (1,2)\})$.

# Paths and cycles

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A *cycle* is a path whose first and last element are equal. We say a graph is *cyclic* if it contains any cycles, and *acyclic* otherwise.

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### Definition

Let $G = (V, E)$ be a graph and $u \in V$. The *neighbourhood of* $u$
$N(u) = \{v \in V \mid (u, v) \in E\}$.

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$$M(G)[u][v] = \begin{cases} 1 & \text{if } (u, v) \in E \\ 0 & \text{otherwise} \end{cases}$$

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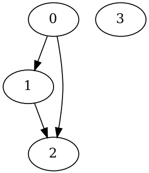


$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Thus, we need a way of determining what our source vertices are.

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However, this is not a complete test — a graph with source vertices may still be cyclic!

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     - 2.2.3 Otherwise, add every $v \in N(u)$ to $C'$
   - 2.3 Set $C = C'$
3. Declare that there are no cycles and stop.

This process is also $O(n^2)$ — we have to make as many checks as there are edges, and there could be as many as $n^2$.

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Thus, we can find both the minimum and maximum parallel width in $O(n^2)$ time as well.

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Thus, we can find the length of the critical path in $O(n^2)$ time as well! We can potentially combine all of these together to avoid traversing the graph more than once.

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However, despite this, we can still use these algorithms in many cases, as long as the number of tasks we're dealing with isn't *too* large.

# Questions?