Hylomorphisms

Or: intermediate structures in a non-awful way

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[AUT CSC]

AUT Computer Science Club

Outline

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Essentially, it's a data structure which we *use*, but never *see*.

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function fib(n)
    if $n = 0$ or $n = 1$ **then**
        return 1
    else
        return fib($n - 1$) + fib($n - 2$)

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function fib$'(n)$
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Since we have a (nearly) n-length loop in $\mathsf{fib}'(n)$, this is clearly $O(n)$ — much better! All thanks to the intermediate array.

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However, it's not all brilliant — intermediate structures have two major issues.

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This is mostly because we’re being forced to consider a specific solution every time we want an intermediate structure to be built up, then torn down.

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Sounds impossible, right?

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Sounds impossible, right? However: there's a morphism for that!

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```
-- our knotting hack
data Knot f = Tie { untie :: f (Knot f) }

-- pronounced `then', for easier composition
>>> :: (a -> b) -> (b -> c) -> (a -> c)
f >>> g = g ∘ f

type Algebra f a = f a -> a

-- catamorphism - tears structures down
cata :: Functor f => Algebra f a -> Knot f -> a
cata f = untie >>> map (cata f) >>> f

type Coalgebra f a = a -> f a

-- anamorphism - builds structures up
ana :: Functor f => Coalgebra f a -> a -> Knot f
ana f = f >>> map (ana f) >>> Tie
```

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Now, `huh` is a very silly name. We should really fix our terminology again...

The humble hylomorphism

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hylo :: Functor f => Coalgebra f a -> Algebra f b -> a -> b
hylo f g = ana f >>> cata g
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The term ‘hylomorphism’ comes from the Greek root *hylo* (meaning ‘tree’), as its behaviour can be seen as tree-like (building up is like the tree growing, and tearing down is like the tree being turned into matches).

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Note: The intermediate structure being operated on by the algebra and coalgebra must be *the same*, but the data can be different.

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  3. When we've finished, return the result
- Implicitly builds up a binary tree of deferred sort operations

The merge procedure, in detail

function merge(a_1, a_2)
  $n \leftarrow \text{len}(a_1) + \text{len}(a_2)$
  $a \leftarrow$ a new array of length n
  $f_1, f_2 \leftarrow 0$
  for $i \in 0, 1, \ldots, n-1$ **do**
    if $f_1 = \text{len}(a_1)$ **then**
      $a[i] = a_2[f_2]$
      $f_2 \leftarrow f_2 + 1$
    else if $f_2 = \text{len}(a_2)$ **then**
      $a[i] \leftarrow a_1[f_1]$
      $f_1 \leftarrow f_1 + 1$
    else
      $e \leftarrow \min\{a_1[f_1], a_2[f_2]\}$
      $a[i] \leftarrow e$
      Increment corresponding f
  return a

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Let's see how this works in practice: we will sort the array `[4, 2, 6, 1]`.

Worked example

$\boxed{\text{sort } [4,2,6,1]}$

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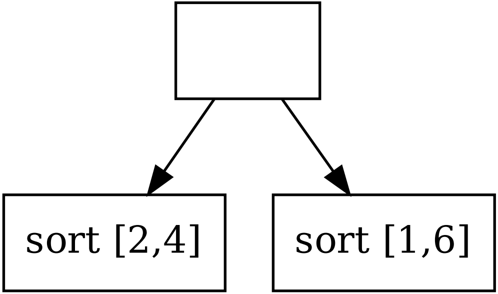

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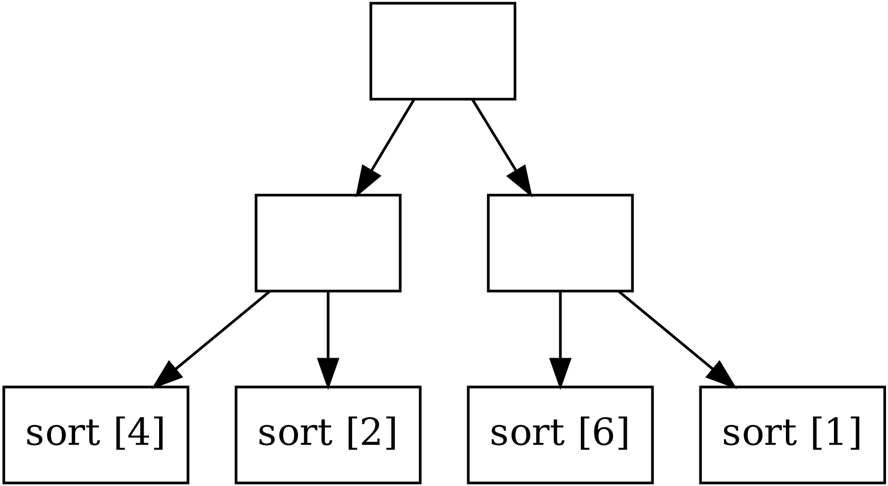

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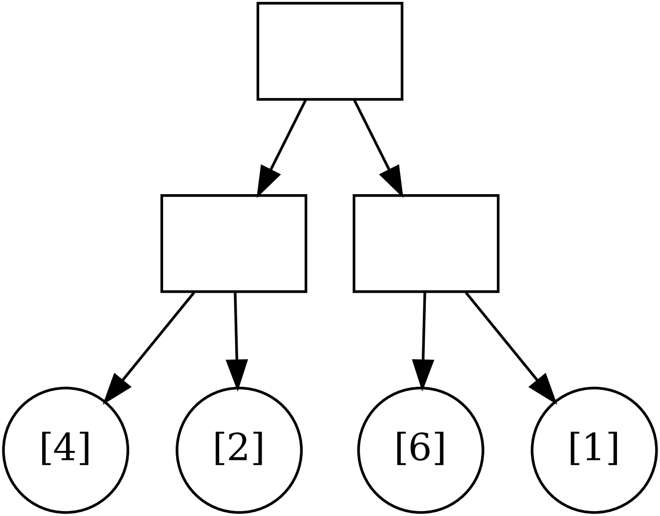

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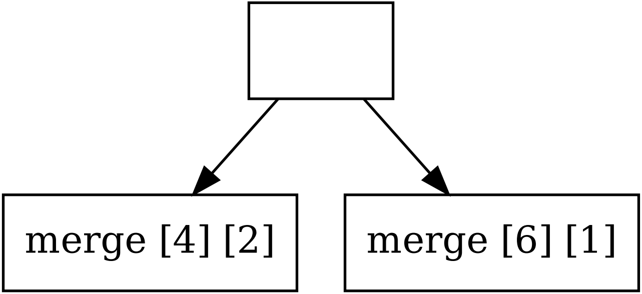

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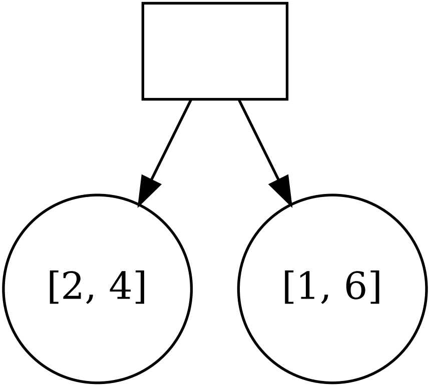

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$$\boxed{\text{merge } [2, 4]\ [1, 6]}$$

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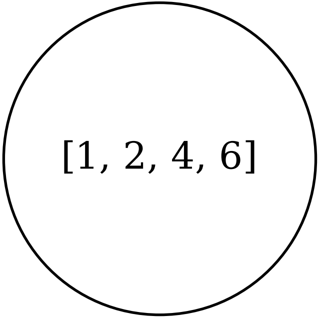

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    Internal (slice v 0 half) (slice v half (x - half))
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Yay!

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Result: We have everything we wanted, and recursion schemes still rule!

Questions?